2013

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be the function $F(x,y) = xy^2 + x - x^3$. Find the critical points of F, find the Hessian of F at each critical point and classify the critical points as maxima, minima or saddle points.

(b) Define the product function $P(x,y,z) = xyz$ and consider its restriction to the ellipsoid $G(x,y,z) = 0$ where

$$G(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

for some constants $0 < a < b < c$. Show that the critical points of P restricted to the ellipsoid are the six points

$$(\pm a, 0, 0),\ (0, \pm b, 0),\ (0, 0, \pm c)$$

and the eight points

$$\frac{1}{\sqrt{3}}(\pm a, \pm b, \pm c)$$

where the $\pm$ signs in the latter case are independent of each other.

(c) Which of these critical points are already critical points of P before restriction to the ellipsoid and which are not? In the latter case, give a geometric interpretation of what is happening in terms of the level sets of P and of G.

1. (Answer) (a) The partial derivatives of F are

$$\frac{\partial F}{\partial x} = y^2 + 1 - 3x^2$$
$$\frac{\partial F}{\partial y} = 2xy.$$

These must both vanish at a critical point, so the critical points are the solutions of

or

$$2xy = 0,\ y^2 + 1 = 3x^2.$$

In particular, either $x = 0$ or $y = 0$. If $x = 0$ then $y^2 + 1 = 0$, but $y^2 \geqslant 0$ so $y^2 + 1 \geqslant 1 > 0$, hence this cannot occur. If $y = 0$ then $3x^2 = 1$ and $x = \pm 1/\sqrt{3}$. The critical points are therefore $(\pm 1/\sqrt{3}, 0)$.

The Hessian of F is its matrix of second derivatives

$$\begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -6x & 2y \\ 2y & 2x \end{pmatrix}.$$

At the critical point $(\pm 1/\sqrt{3}, 0)$ this matrix is

$$\begin{pmatrix} \mp 6/\sqrt{3} & 0 \\ 0 & \pm 2/\sqrt{3} \end{pmatrix}.$$

Since the Hessian has one positive and one negative eigenvalue at each critical point, these critical points are saddle points.

(b) We introduce a Lagrange multiplier and consider the modified functional

$$H(x,y,z,\lambda) = P(x,y,z) - \lambda G(x,y,z) = xyz - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

The equations for vanishing of the first derivatives of H with respect to its four variables are

$$yz - 2\lambda x/a^2 = 0$$
$$xz - 2\lambda y/b^2 = 0$$
$$xy - 2\lambda z/c^2 = 0$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The first three imply

$$2\lambda = yza^2/x = xzb^2/y = xyc^2/z$$

so

$$zy^2a^2 = zx^2b^2,\ xz^2b^2 = xy^2c^2,\ yz^2a^2 = yx^2c^2$$

We now start case analysis. If $x, y, z \neq 0$ then the first equation gives $ya = \pm xb$, the second gives $zb = \pm yc$ and the third gives $az = \pm cx$. Therefore $y = \pm xb/a$ and $z = \pm cx/a$. The ellipsoid constraint implies

$$x^2(1/a^2 + 1/a^2 + 1/a^2) = 1$$

therefore $x = \pm a/\sqrt{3}$. Together with the expressions for y and z in terms of x this gives the eight listed critical points.

If $x = 0$ then $yz = 2\lambda x/a^2 = 0$ so either $y = 0$ or $z = 0$ too. In this way we see that any more critical points have precisely two coordinates vanishing and when two coordinates vanish we get a solution with $\lambda = 0$. This gives the remaining six critical points.

(c) The critical points of P have $xy = xz = yz = 0$ so they are points where at least two of the coordinates vanish. The six critical points $(\pm a, 0, 0), (0, \pm b, 0), (0, 0, \pm c)$ are of this type. The other critical points occur when the level sets of P are tangent to the ellipsoid $\{G = 0\}$.

2. Consider the quasilinear first-order partial differential equation

$$x^2\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial y}+y=0.$$

(a) Write down the characteristic vector field and show that its integral curves are

$$(x(t),y(t),z(t))=\left(-\frac{1}{t+K},M\cos t,-M\sin t\right)$$

where K and M are constants of integration.

(b) If we impose the initial condition $\phi(1,s)=s$, show that the solution surface can be parametrised as

$$(s,t)\mapsto\left(-\frac{1}{t-1+\frac{\pi}{4}},s\sqrt{2}\cos t,-s\sqrt{2}\sin t\right). \qquad (*)$$

(c) The caustic of a surface

$$(s,t)\mapsto(x(s,t),y(s,t),z(s,t))$$

is the set of values (x,y) where

$$\det\begin{pmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{pmatrix}=0.$$

What is the caustic of the solution surface (*)?

(d) Find a function $\phi(x,y)$, defined on a subset of $\mathbf{R}^2$, such that the graph of ϕ coincides with part of the solution surface. Remember to specify the domain of definition of your function.

2. (Answer) (a) The characteristic vector field in $\mathbf{R}^3$ with coordinates (x,y,z) is

$$(x^2,z,-y)$$

The characteristic curves $(x(t),y(t),z(t))$ solve $\dot{x}=x^2$, $\dot{y}=z$, $\dot{z}=-y$. Hence $\int dx/x^2=\int dt$ and $-1/x=t+K$ so $x=-1/(t+K)$. Also $\ddot{y}=-y$ and $\ddot{z}=-z$ so $y=M\cos t+N\sin t$ and $z=-M\sin t+N\cos t$. By reparametrising t we can take $N=0$. This gives the characteristic curves in the form required.

(b) Imposing $\phi(1,s)=s$ gives

$$1=-1/(t+K),\quad s=M\cos t=-M\sin t$$

so $\tan t=-1$ and $t=-\pi/4+n\pi$ so that $K=-1-n\pi+\pi/4$. If we reparametrise t to $t-n\pi$ then we can assume $K=-\pi/4$ and the sign change in $\cos(t+n\pi)$ and $\sin(t+n\pi)$ can be absorbed into s. Therefore $M=s\sqrt{2}$ and the solution surface is

$$(s,t)\mapsto\left(\frac{-1}{t-1+\pi/4},s\sqrt{2}\cos t,-s\sqrt{2}\sin t\right).$$

(c) The determinant we need to compute is

$$\det\begin{pmatrix}0 & 1/(t-1+\pi/4)^2\\ \sqrt{2}\cos t & -s\sqrt{2}\sin t\end{pmatrix}=\pm\frac{\sqrt{2}\cos t}{t-1+\pi/4}$$

This vanishes if and only if $\cos t=0$, i.e. $y=0$ and $t=(m+1/2)\pi$ for some $m\in\mathbf{Z}$. Therefore the caustic consists of the points with $y=0$ and $x=1/(3\pi/4+m\pi-1)$.

(d) If $x \neq 0$ then $x = -1/(t - 1 + \pi/4)$ so $t = 1 - \pi/4 - 1/x$. Then

$$z = -s\sqrt{2}\sin t = -y\tan t = -y\tan\left(1 - \pi/4 - \frac{1}{x}\right)$$

i.e.

$$\phi(x, y) = -y\tan\left(1 - \pi/4 - \frac{1}{x}\right)$$

This function is well-defined when $x \neq 0$ and when $1 - \pi/4 - 1/x \neq (m + 1/2)\pi$, i.e. $x \neq 1/(1 - m\pi - 3\pi/4)$.

3. Suppose that a function $y(x)$ satisfies the Euler-Lagrange equation for a functional $\int_a^b L(x, y, y')dx$ such that the Lagrangian L has no explicit dependence on its first variable x. Prove the *Beltrami identity*:

$$L - y'\frac{\partial L}{\partial y'} = \text{constant}.$$

Consider the space of functions $y(x)$

$$X = \{y\colon [0,1] \to \mathbf{R} \ : \ y(0) = 0, \ y(1) = 0\}.$$

Define the functionals

$$F(y) = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and

$$G(y) = \int_0^1 (y - 1)dx.$$

By extremising a suitably modified functional, show that if $y(x)$ is a critical point of F restricted to the set $\{G(y) = 0\}$ then the graph of y is a segment of a circle. *You do not need to compute any of the constants of integration or the Lagrange multiplier.*

What would the graph of a critical point be if we did not impose the restriction $G(y) = 0$?

3. (Answer) First of all we note that if the Lagrangian L has no explicit x-dependence and

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$$

then

$$L - y'\frac{\partial L}{\partial y'}$$

is constant (this is Beltrami's identity). To prove this, differentiate

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial x} + y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'} - y''\frac{\partial L}{\partial y'} - y'\frac{d}{dx}\frac{\partial L}{\partial y'}$$

which vanishes if $\partial L/\partial x = 0$ and the Euler-Lagrange equation holds.

The functional we need to consider for our problem is

$$F(y) - \lambda G(y) = \int_0^1 (\sqrt{1 + (y')^2} - \lambda(y - 1))dx$$

where λ is a Lagrange multiplier. Therefore $\quad$ 1 or

$$L = \sqrt{1 + (y')^2} - \lambda(y - 1)$$

The Beltrami identity gives:

$$\sqrt{1 + (y')^2} - \lambda(y - 1) - \frac{(y')^2}{\sqrt{1 + (y')^2}} = C$$

for some constant C. This simplifies to

$$C = \frac{1}{\sqrt{1 + (y')^2}} - \lambda(y - 1)$$

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which gives

$$y' = \sqrt{\frac{1}{(C+\lambda(y-1))^2} - 1}$$

or, defining $C - \lambda = D$

$$\frac{(D+\lambda y)y'}{\sqrt{1-(D+\lambda y)^2}} = 1$$

Integrating using the subtitution $\cos\theta = \lambda y + D$ gives

$$(D+\lambda y)^2 + (E+\lambda x)^2 = 1$$

so that the graph of y is a segment of a circle.

If we did not use the constraint $G(y) = 0$ we would end up with a segment of a straight line connecting the two points $(0,0)$ and $(1,0)$. To see this, note that the new Beltrami identity is

$$\sqrt{1+(y')^2} - (y')^2/\sqrt{1+(y')^2} = C$$

and this rearranges to

$$1 + (y')^2 - (y')^2 = C\sqrt{1+(y')^2}$$

or

$$y' = \sqrt{1/C^2 - 1}.$$

Since y' is constant, y must be linear and hence its graph is a straight line segment joining the endpoints which are fixed by the boundary conditions.

4. (a) Suppose that $\phi\colon [0,\pi] \times [0,\pi] \to \mathbf{R}$ solves Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

By separating variables and solving the resulting ordinary differential equations, derive expressions for the separated solutions $\phi(x,y) = X(x)Y(y)$.

(b) We impose the boundary conditions

$$\phi(x,0) = 0,\ \phi(x,\pi) = e^x - 1,\ \phi(y,0) = 0,\ \phi(y,\pi) = e^y - 1.$$

Show that the solution ϕ is

$$\phi(x,y) = \frac{xy(e^\pi - 1)}{\pi^2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{e^\pi(-1)^n - 1}{n(n^2+1)}(\sin nx \underbrace{\sinh ny}_{\sinh n\pi} + \underbrace{\sinh nx}_{\sinh n\pi} \sin ny).$$

Hint: It may help you when computing Fourier series to remember that

$$\sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}).$$

4. (Answer) (a) Separating variables in Laplace's equation yields the ordinary differential equation

$$X''Y + XY'' = 0$$

which implies

$$X''/X = -Y''/Y = \lambda$$

where λ is a constant because X''/X depends only on x and Y''/Y depends only on y. The analysis separates into three cases:

- $\lambda = 0$ which gives $X(x) = Ax + B$ and $Y(y) = Cy + D$.
- $\lambda = p^2 > 0$ which gives $X(x) = A\cosh(px) + B\sinh(px)$ and $Y(y) = C\cos(py) + D\sin(py)$.
- $\lambda = -p^2 < 0$ which gives $X(x) = A\cos(px) + B\sin(px)$ and $Y(y) = C\cosh(py) + D\sinh(py)$.

(b) If we impose the boundary conditions

$$\phi(x,0) = \sin x,\ \phi(x,\pi) = e^x - 1,\ \phi(y,0) = 0,\ \phi(y,\pi) = e^y - 1$$

then we have $\phi(\pi,\pi) = e^\pi - 1$ and ϕ vanishes at all the other corners. The $\lambda = 0$-separated solution

$$\phi_0(x,y) = (e^\pi - 1)xy/\pi^2$$

has the same corner values and hence if ϕ is a solution then $\theta = \phi - \phi_0$ vanishes at the corners. This new function satisfies the boundary conditions

$$\theta(x,0) = 0,\ \theta(x,\pi) = e^x - 1 - x(e^\pi - 1)/\pi,\ \theta(y,0) = 0,\ \theta(y,\pi) = e^y - 1 - y(e^\pi - 1)/\pi.$$

We will solve the two nontrivial boundary problems separately and add the solutions afterwards invoking linearity of the Laplace equation to see that the result is still a solution.
The first problem is $\theta_2(x,\pi) = e^x - 1 - x(e^\pi - 1)/\pi$ and θ_2 vanishes on all other boundary components. We must compute the Fourier series

$$e^x - 1 - x(e^\pi - 1)/\pi = \sum_{n=1}^{\infty} A_n \sin nx$$

and we see that the solution will be

$$\sum_{n=1}^{\infty} A_n \sin nx \frac{\sinh ny}{\sinh n\pi}.$$

To compute the Fourier series we expand $\sin nx = \frac{e^{inx}-e^{-inx}}{2i}$ and compute

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{e^{(1+in)x} - e^{(1-in)x}}{2i}dx - \frac{2(e^{\pi}-1)}{\pi^2}\int_0^{\pi} x\sin nx dx - \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx$$

The first integral gives

$$\frac{2}{\pi(1+in)2i}\left(e^{(1+in)\pi}-1\right) - \frac{2}{\pi(1-in)2i}\left(e^{(1-in)\pi}-1\right)$$

or

$$\frac{1}{\pi}\left(\frac{(-1)^n e^{\pi}-1}{i-n} - \frac{(-1)^n e^{\pi}-1}{i+n}\right) = \frac{(1-(-1)^n e^{\pi})}{\pi}\frac{2n}{n^2+1}$$

and the second integral gives

$$-\frac{2(e^{\pi}-1)}{\pi^2}\int_0^{\pi} x\sin nx dx = \left[\frac{2(e^{\pi}-1)}{n\pi^2}x\cos nx\right]_0^{\pi} - \frac{2(e^{\pi}-1)}{n\pi}\int_0^{\pi}\cos nx dx$$
$$= \frac{2(e^{\pi}-1)}{n\pi}(-1)^n$$

and the third gives

$$\frac{-2}{\pi}\int_0^{\pi}\sin nx dx = \frac{2}{n\pi}[\cos nx]_0^{\pi} = \frac{2}{n\pi}((-1)^n-1)$$

so

$$A_n = \frac{(1-(-1)^n e^{\pi})}{\pi}\frac{2n}{n^2+1} + \frac{2(e^{\pi}-1)}{n\pi}(-1)^n + \frac{2}{n\pi}((-1)^n-1)$$

which simplifies to

✓OR

$$A_n = \frac{2}{\pi}\frac{e^{\pi}(-1)^n-1}{n(n^2+1)}.$$

The second problem is the same as the first but with x and y swapped, so the solution is

$$\theta_3(x,y) = \sum_{n=1}^{\infty} A_n \sin ny \frac{\sinh nx}{\sinh n\pi}.$$

The total solution is therefore $\phi_0 + \theta_2 + \theta_3$ where the various terms are defined above.

5. What does it mean for a quasilinear partial differential equation of second order to be elliptic, parabolic or hyperbolic? Verify that the minimal surface equation

$$\frac{\partial^2\varphi}{\partial x^2}\left(1+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)+\frac{\partial^2\varphi}{\partial y^2}\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)=2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

is elliptic.

Consider the wave equation

$$\frac{\partial^2\phi}{\partial t^2}=c(x)^2\frac{\partial^2\phi}{\partial x^2}$$

where $c(x)\neq 0$ may depend on x. Show that this equation is hyperbolic.

Suppose that c in the wave equation depends discontinuously on x:

$$c(x)=\begin{cases}1 & \text{if } x<0\\ 2 & \text{if } x\geqslant 0\end{cases}$$

Find the separated solutions $\phi(x,t)=X(x)T(t)$ to this equation on an infinite string which are oscillatory (i.e. trigonometric) in time and such that $\phi(x,t)$ and $\partial\phi/\partial x(x,t)$ are continuous at $x=0$.

Suppose that $\phi(x,t)=\sin x\sin t$ on $x<0$. What is $\phi(x,t)$ on $x>0$?

5. (Answer) A quasilinear partial differential equation of second order in two variables is an equation of the form

$$A\left(x,y,\phi,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x^2}+B\left(x,y,\phi,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}+C\left(x,y,\phi,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial y^2}=0$$

and it is called hyperbolic/parabolic/elliptic according to whether the discriminant

$$\underline{B^2-4AC}$$

(a function depending on the five quantities $x,y,\phi,\partial\phi/\partial x,\partial\phi/\partial y$) is positive-, zero- or negative-valued respectively.

In the case of the minimal surface equation

$$\frac{\partial^2\varphi}{\partial x^2}\left(1+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)+\frac{\partial^2\varphi}{\partial y^2}\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)=2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

we have

$$A=\left(1+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)$$

$$B=-2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}$$

$$C=\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)$$

so the discriminant is

$$4\left(\frac{\partial\varphi}{\partial x}\right)^2\left(\frac{\partial\varphi}{\partial y}\right)^2-4\left(1+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)$$

or

$$-4\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)<0$$

which shows that the equation is elliptic.

The wave equation

$$\frac{\partial^2\phi}{\partial t^2}=c(x)^2\frac{\partial^2\phi}{\partial x^2}$$

has $A=1$, $B=0$, $C=-c(x)^2$ and so $B^2-4AC=4c(x)^2>0$ which means the equation is hyperbolic.

The separated solutions for the wave equation with the given function $c(x)$ are

$$\phi(x,t)=\begin{cases}X_1(x)T_1(t) & \text{if } x<0\\ X_2(x)T_2(t) & \text{if } x\geqslant 0\end{cases}$$

with

$$X_1''T_1=X_1T_1'',\ 4X_2''T_2=X_2T_2''$$

or

$$X_1''=\lambda X_1,\ T_1''=\lambda T_1, X_2''=\frac{\mu}{4}X_2'',\ T_2''=\mu T_2.$$

If ϕ and $\partial\phi/\partial x$ are to be continuous at $x=0$ we need $\lambda=\mu$ otherwise we would get a nontrivial linear dependence between (linearly independent) hyperbolic or trigonometric functions of t with frequency $\sqrt{\pm\mu}$ and frequency $\sqrt{\pm\lambda}$ by setting $X_1(0)T_1(t)=X_2(0)T_2(t)$ or $X_1'(0)T_1(t)=X_2'(0)T_2(t)$. Note that if both of these relationships were trivial (i.e. if $X_1(0)=X_1'(0)$ and $X_2'(0)=X_2(0)=0$) then we would get $X_1\equiv 0$ and $X_2\equiv 0$ from the fact that X_1 and X_2 satisfy linear second order equations.

Therefore we have three cases according to whether $\lambda>0,=0,<0$ and since we are looking for oscillatory solutions we can restrict to $\lambda=-p^2<0$. In this case $X_1(x)=A_1\sin(px))+B_1\cos(px)$ and $T_1(t)=C_1\sin(pt)+D_1\cos(pt)$, $X_2(x)=A_2\sin(px/2)+B_2\cos(px/2)$ and $T_2(t)=C_2\sin(pt)+D_2\cos(pt)$. Continuity of ϕ at $x=0$ implies

$$B_1\left(C_1\sin(pt)+D_1\cos(pt)\right)=B_2\left(C_2\sin(pt)+D_2\cos(pt)\right)$$

which means that $B_1C_1=B_2C_2$, $B_1D_1=B_2D_2$. Continuity of $\partial\phi/\partial x$ at $x=0$ implies

$$pA_1\left(C_1\sin(pt)+D_1\cos(pt)\right)=\frac{p}{2}A_2\left(C_2\sin(pt)+D_2\cos(pt)\right)$$

which means that $2A_1C_1=A_2C_2$ and $2A_1D_1=A_2D_2$.

The general oscillatory separated solution which is continuous and has continuous first derivatives at $x=0$ is therefore

$$\phi(x,t)=\begin{cases}A\sin px\sin pt+B\sin px\cos pt+C\cos px\sin pt+D\cos px\cos pt & \text{if } x<0\\ 2A\sin\left(\frac{px}{2}\right)\sin pt+2B\sin\left(\frac{px}{2}\right)\cos pt+C\cos\left(\frac{px}{2}\right)\sin pt+D\cos\left(\frac{px}{2}\right)\cos pt & \text{if } x\geqslant 0\end{cases}$$

If $X_1(x)T_1(t)=\sin x\sin t$ on $x<0$ then the solution on $x\geqslant 0$ is $2\sin(x/2)\sin t$.

6. State coordinates u and v such that

$$3\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y^2} = C\frac{\partial^2\phi}{\partial u\partial v} \qquad (1)$$

for some constant C, and find the constant C. Check explicitly that Equation (1) holds for the coordinates you have chosen.

Sketch the lines $u =$ constant and $v =$ constant on a spacetime diagram.

Find the general solution of the equation

$$3\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y^2} = \frac{16}{9}(x^2 - 2xy - 3y^2).$$

If we are given that

$$\phi(x,0) = \cos x,\ \frac{\partial\phi}{\partial y}(x,0) = e^x,$$

what is ϕ?

6. (Answer) Consider the quadratic polynomial

$$3s^2 - 2s - 1 = 0$$

Its roots are $s_\pm = -1/3,\ 1$ and the coordinates we need are

$$u = y + x,\ v = y - x/3$$

We check that

$$y = \frac{1}{4}(3v + u),\ x = \frac{3}{4}(u - v)$$

so using the chain rule

$$\begin{aligned}
\frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} \\
&= \frac{1}{4}\left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial v} &= \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} \\
&= \frac{3}{4}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)
\end{aligned}$$

Therefore

$$\partial^2\phi/\partial u\partial v = \frac{3}{16}(-3\partial_x^2 + 2\partial_x\partial_y + \partial_y^2)$$

which is $-3/16$ times the desired operator.

The lines we must sketch are $x + y = C$ and $y - x/3 = C$, that is straight lines with slope -1 and $1/3$.

SKETCH

We have

$$-1/3(x^2 - 2xy - 3y^2) = uv$$

so

$$-(16/3)\partial_u\partial_v\phi = 3\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y^2} = \frac{16}{9}(x^2 - 2xy - 3y^2) = -(16/3)uv$$

or

$$\partial_u\partial_v\phi = uv.$$

Integrating with respect to u and v gives

$$\phi = u^2v^2/4 + F(u) + G(v)$$

for some arbitrary functions F and G. This is the general solution. In terms of x and y it is

$$\phi(x, y) = (x+y)^2(y - x/3)^2/4 + F(x+y) + G(y - x/3).$$

If we are given that

$$\phi(x, 0) = \cos x, \; \frac{\partial\phi}{\partial y}(x, 0) = e^x,$$

then we know

$$x^4/36 + F(x) + G(-x/3) = \cos x$$

and, since

$$\partial\phi/\partial y = (x+y)(y-x/3)(y - x/3 + x + y)/2 + F'(x+y) + G'(y - x/3),$$

we have

$$-x^3/9 + F'(x) + G'(-x/3) = e^x.$$

Differentiating the first of these gives

$$x^3/9 + F'(x) - G'(-x/3)/3 = -\sin x$$

and using the two as simultaneous equations for F' and G' we get

$$4F'(x) = e^x - 3\sin x - 2x^3/9$$

so

$$F(x) = -\frac{x^4}{72} + \frac{e^x}{4} + \frac{3}{4}\cos x$$

and hence

$$G(-x/3) = -\frac{x^4}{72} + \frac{1}{4}(\cos x - e^x)$$

or, substituting $z = -x/3$,

$$G(z) = -\frac{81z^4}{72} + \frac{1}{4}(\cos 3z - e^{-3z}).$$

Therefore the solution is

$$\begin{aligned}\phi(x, y) &= (x+y)^2(y - x/3)^2/4 + F(x+y) + G(y - x/3)\\ &= (x+y)^2(y-x/3)^2/4 - \frac{(x+y)^4}{72} + \frac{e^{x+y}}{4} + \frac{3}{4}\cos(x+y)\\ &\quad - \frac{81(y - x/3)^4}{72} + \frac{1}{4}(\cos 3(y - x/3) - e^{-3(y-x/3)}).\end{aligned}$$